# Tutorial 2

These problems explore the quantization and correlators of metric perturbations during inflation. Notice that the problems indicated as "**Optional**" should be tackled last.

1. *Scale invariance*

   The fact that the de Sitter metric,

$$ds^2 = \frac{-d\tau^2 + dx^i \delta_{ij} dx^j}{\tau^2 H^2}\,, \tag{1}$$

   is invariant under dilations, $\{\tau, \mathbf{x}\} \to \{\lambda\tau, \lambda\mathbf{x}\}$, implies that any correlator which does not depend explicitly on time must obey,

$$\langle \phi(\mathbf{x}_1) \dots \phi(\mathbf{x}_n) \rangle = \langle \phi(\lambda\mathbf{x}_1) \dots \phi(\lambda\mathbf{x}_n) \rangle\,. \tag{2}$$

   Show that the corresponding momentum-space correlator,

$$\langle \phi_{\mathbf{k}_1} \dots \phi_{\mathbf{k}_n} \rangle =: (2\pi)^3 \delta^3 \left( \sum_{a=1}^{n} \mathbf{k}_a \right) B_n(\mathbf{k}_1, \dots, \mathbf{k}_n)\,, \tag{3}$$

   must therefore scale as,

$$B_n(\lambda\mathbf{k}_1, \dots, \lambda\mathbf{k}_n) = \frac{1}{\lambda^{3(n-1)}} B_n(\mathbf{k}_1, \dots, \mathbf{k}_n)\,. \tag{4}$$

2. *My first wavefunction*

   In this problem you will first compute the cubic wavefunction coefficient and then use it to compute the bispectrum (three-point function).

   (a) Using the appropriate diagrammatic Feynman rules, compute the cubic wavefunction coefficient $\psi_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ for a massless scalar field on de Sitter induced by the interactions $\dot{\varphi}^3$ and $\dot{\varphi}(\partial_i \varphi)^2$.

   (b) Using the Born rule derive the following relation between the cubic wavefunction coefficient $\psi_3$ and the bispectrum,

$$\langle \prod_{a=1}^{3} \phi(\mathbf{k}_a) \rangle' = -\frac{\mathrm{Re}\, \psi_3(\{\mathbf{k}\})}{\prod_{a=1}^{3} \mathrm{Re}\, \psi_2(k_a)}\,. \tag{5}$$

   which is valid to linear order in $\psi_3$ and for parity-even interactions.

   (c) Compare the above result with the direct calculation you did in Problem 4 of the first tutorial.

3. *Check of the cosmological optical theorem*

   In this problem you will check the cosmological optical theorem for the tree-level quartic wavefunction coefficient $\psi_4$ of a scalar in Minkowski.

   (a) Using the Feynman rules for the wavefunction and the Minkowski propagators

$$K_k(t) = e^{iEt} \qquad G_k(t_1, t_2) = -i \frac{e^{iEt_2}}{E} \sin(Et_1) \theta(t_1 - t_2) + (t_1 \leftrightarrow t_2)\,, \tag{6}$$

compute the tree-level contact $\psi_3$ and **optionally** the $s$-channel tree-level exchange $\psi_4$ from the interaction $\lambda\phi^3/3!$ in Minkowski spacetime. You should find

$$\psi_3 = -\frac{\lambda}{E_1+E_2+E_3}\,, \qquad \psi_{4,s} = \frac{\lambda^2}{E_L E_R E_T}\,, \qquad P = \frac{1}{2E} \tag{7}$$

where $E = \sqrt{k^2+m^2}$ and

$$E_L \equiv E_1 + E_2 + E_s\,, \qquad E_R \equiv E_3 + E_4 + E_s\,, \qquad E_T = \sum_a^4 E_a\,. \tag{8}$$

(b) Hence, check that the following relation implied by unitarity is satisfied if $\lambda \in \mathbb{R}$,

$$\begin{aligned}\psi_{4,s}(E_a,s) + \psi_{4,s}(-E_a,s)^* = &-P(s)\left[\psi_3(E_1,E_2,s) + \psi_3(-E_1,-E_2,s)^*\right] \\ &\times\left[\psi_3(E_3,E_4,s) + \psi_3(-E_3,-E_4,s)^*\right]\end{aligned} \tag{9}$$

4. *Check the manifestly local test*

   (a) The cubic wavefunction coefficient corresponding to the interactions $\dot\phi^3$ was computed in Problem 2 and is

   $$\psi_3 \propto \frac{(k_1k_2k_3)^2}{E_T^3}\,, \tag{10}$$

   where $E_T = k_1+k_2+k_3$. Verify that this satisfies the manifestly local test

   $$\partial_{k_1}\psi_3\big|_{k_1=0} = 0\,, \tag{11}$$

   (b) The cubic wavefunction coefficient corresponding to the interactions $\dot\phi\partial_i\phi^2$ was computed in Problem 2. Let's write it as

   $$\begin{aligned}\psi_3 \propto \frac{1}{E_T^3}\Bigg[&24\,(k_1k_2k_3)^2 - 8E_T\,(k_1k_2k_3)\left(\sum_{a<b}k_ak_b\right) \\ &- C_1\,E_T^2\left(\sum_{a<b}k_ak_b\right)^2 + 22\,E_T^3\,(k_1k_2k_3) - C_2E_T^4\left(\sum_{a<b}k_ak_b\right) + C_3E_T^6\Bigg]\,,\end{aligned} \tag{12}$$

   where $C_{1,2,3}$ are three numerical constants. Using the manifestly local test determine $C_{1,2,3}$. You should find $C_1 = 8$, $C_2 = 6$ and $C_3 = 2$.

5. **Optional:** *Tensor power spectrum.*

   Using

   $$S_2 = \frac{M_P^2}{8}\int d^3x d\tau\, a^2\left[\gamma'_{ij}\gamma'_{ij} - \partial_i\gamma_{jk}\partial_i\gamma_{jk}\right] \quad \text{(on de Sitter)}, \tag{13}$$

   derive the amplitude of the tensor power spectrum.

6. **Optional:** *De Sitter Ward identities.*

   De Sitter spacetime has 10 isometries (the same number as Minkowski). We have already seen that the 3 spatial translations and 3 rotations (homogeneity and isotropy), as well as the 1 scaling isometry ($\tau \to \lambda\tau, \mathbf{x}^i \to \lambda\mathbf{x}^i$), impose constraints on the correlators. The other 3 isometries are given by,

   $$\begin{gathered}\tau \to \gamma\,\tau \quad , \quad \mathbf{x}^i \to \gamma\left(\mathbf{x}^i + \mathbf{b}^i\,(\tau^2 - |\mathbf{x}|^2)\right) \\ \text{where} \quad \gamma = \left(1 - 2\mathbf{b}\cdot\mathbf{x} - |\mathbf{b}|^2(\tau^2 - |\mathbf{x}|^2)\right)^{-1}\end{gathered} \tag{14}$$

   and are the analogue of *boosts* on de Sitter. These symmetries also impose non-trivial constraints on the correlators, which you will now find.

(a) Consider the infinitesimal symmetry transformation (expand (14) at small $|\mathbf{b}|$). Show that this symmetry is generated by the operator,

$$\hat{\mathbf{K}}_i[\tau, \mathbf{x}] = -2\mathbf{x}_i \left(\tau \partial_\tau + \mathbf{x} \cdot \partial_{\mathbf{x}}\right) + (\tau^2 - |\mathbf{x}|^2) \partial_{\mathbf{x}^i} \, . \tag{15}$$

(b) By transforming carefully to momentum space, show that the corresponding momentum space operator is,

$$\hat{\mathbf{K}}_i[\tau, \mathbf{k}] = 2\left(d - \tau \partial_\tau + \mathbf{k} \cdot \partial_{\mathbf{k}}\right) \partial_{\mathbf{k}} - \mathbf{k}\left(\tau^2 + \partial_{\mathbf{k}}^2\right) \, , \tag{16}$$

where $d = 3$ is the number of spatial dimensions.

(c) Hence derive the corresponding Ward identity for cosmological correlators,

$$\sum_{b=1}^{n} \hat{\mathbf{K}}_i[\tau_b, \mathbf{k}_b] \langle \hat{\varphi}_{\mathbf{k}_n}(\tau_n) ... \hat{\varphi}_{\mathbf{k}_1}(\tau_1) \rangle = 0 \tag{17}$$

(d) For a massless scalar field, $\partial_\tau \varphi_{\mathbf{k}} \sim \tau \to 0$ in the limit $\tau \to 0$, and so (17) becomes,

$$\sum_{b=1}^{n} \left[ 2(d + \mathbf{k}_b \cdot \partial_{\mathbf{k}_b}) \partial_{\mathbf{k}_b} - \mathbf{k}_b \partial_{\mathbf{k}_b}^2 \right] \langle \varphi_{\mathbf{k}_n} ... \varphi_{\mathbf{k}_1} \rangle = 0. \tag{18}$$

for the equal-time in-in correlator at the end of inflation. Show that when the correlator is a function of the magnitudes $k_1, ... k_n$ only, then this simplifies to,

$$0 = \left(K_b - K_{b'}\right) \langle \varphi_{\mathbf{k}_n} ... \varphi_{\mathbf{k}_1} \rangle \tag{19}$$

$$\text{where} \quad K_b = \frac{d+1}{k_b} \partial_{k_b} + \partial_{k_b}^2 \, .$$

for any pair of fields $(b, b')$.

*Hint. You may use the fact that translation invariance implies that the total momentum must vanish, so* $\mathbf{k}_n = -\sum_{b=1}^{n-1} \mathbf{k}_b$.

(e) Check whether the bispectrum $\langle \varphi_{\mathbf{k}_3} \varphi_{\mathbf{k}_2} \varphi_{\mathbf{k}_1} \rangle$ computed in the lectures from the interaction $\int d^4 \sqrt{-g}\, \varphi^3$,

$$\langle \varphi_{\mathbf{k}_1} \varphi_{\mathbf{k}_2} \varphi_{\mathbf{k}_3} \rangle \propto \frac{1}{(k_1 k_2 k_3)^2} \left[ 4 - \sum_{b,c} \frac{k_b}{k_c} + \frac{\sum_b k_b^3}{k_1 k_2 k_3} \log(k_T \tau) \right] \tag{20}$$

satisfies the (dS) boost Ward identity with the bulk operator in (16). Do you expect that the bispectra from $\dot{\varphi}^3$ and $\dot{\varphi}(\partial_i \varphi)^2$ that you computed in examples sheet 1 will satisfy these dS Ward identities?

*Hint. Set* $Z = c_s = 1$ *and think carefully about what answer you expect to find in each case before starting.*

(f) Compute the bispectrum from the interaction,

$$S_3 = \int d^4 x \sqrt{-g}\, \varphi (\nabla_\mu \varphi)^2 \tag{21}$$

for a massless scalar $\varphi$ on de Sitter and confirm that when expanded at small $\tau$ it satisfies the Ward identity (18).

*Hint. There is a trick to doing the in-in time integral for this particular interaction: try integrating by parts to make a factor of* $\Box f_k(\tau)$*, which must vanish since the mode functions obey the classical equations of motion.*

7. **Optional:** *Momentum conservation*

The fact that an FLRW background is invariant under translations, $\mathbf{x} \to \mathbf{x} + \mathbf{b}$, implies that correlators must also be invariant

$$\langle \phi(\mathbf{x}_1) \dots \phi(\mathbf{x}_n) \rangle = \langle \phi(\mathbf{x}_1 + \mathbf{b}) \dots \phi(\mathbf{x}_n + \mathbf{b}) \rangle \,. \tag{22}$$

Using this, prove that momentum-space correlators must always be proportional to a delta function of the total momentum

$$\langle \phi_{\mathbf{k}_1} \dots \phi_{\mathbf{k}_n} \rangle \propto \delta^3 \left( \sum_{a=1}^{n} \mathbf{k}_a \right) \,. \tag{23}$$