Tutorial 1

These problems explore the dynamics and correlators of a scalar field on an inflating spacetime background. Notice that the problems indicated as "**Optional**" should be tackled last.

1. *Classical dynamics of a massless scalar in FLRW*

   Consider a canonical (massless) scalar field,

$$S[\varphi] = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi \right] . \tag{1}$$

   (a) Find the classical equation of motion for φ.

   (b) **Optional**: compute the stress-energy tensor $T^{\mu\nu}$ and show that $\nabla_\mu T^{\mu\nu} = 0$ is a consequence of the equations of motion.

   (c) For an FLRW background, show that the equation of motion in conformal time becomes,

$$\partial_\tau^2 (a\varphi) + \left(k^2 - \frac{a''}{a} \right) (a\varphi) = 0 , \tag{2}$$

   where $a' := \partial_\tau a$. This is the equation of a damped harmonic oscillator.

2. *The massless de Sitter mode functions*

   Here you will derive the mode functions f_k appearing in the quantum free field as

$$\hat{\varphi}_{\mathbf{k}}(\tau) = f_k(\tau) \hat{a}_{\mathbf{k}} + f_k^*(\tau) \hat{a}^\dagger_{-\mathbf{k}} . \tag{3}$$

   (a) Start from the Minkowski mode functions

$$f_k^{\text{Mink}}(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau} \quad . \tag{4}$$

   Show that at sufficiently early times the de Sitter eom becomes the Minkowski eom for af_k. We therefore expect af_k to behave like (4) at early times.

   (b) The most general solution to the de Sitter eom is,

$$f_k(\tau) = \alpha(1 + ik\tau) e^{-ik\tau} + \beta(1 - ik\tau) e^{+ik\tau} , \tag{5}$$

   where α and β are constants of integration. Find the α and β required for af_k and $\partial_\tau(af_k)$ to match f_k^{Mink} and $\partial_\tau f_k^{\text{Mink}}$ at some reference time τ_*.

   (c) Finally, show that in the limit $\tau_* \to -\infty$ these coefficients obey,

$$\alpha \to i\sqrt{\frac{H^2}{2k^3}} \quad , \quad \beta = i\sqrt{\frac{H^2}{2k^3}} \frac{e^{-2ik\tau_*}}{2(k\tau_*)^2} \to 0 \tag{6}$$

   and you therefore recover the mode function derived in the lectures (up to an unimportant overall phase).

3. *Two-point correlators in free theories*

   Using the Heisenberg picture, show that for a free massless scalar field on de Sitter,

   (a)
$$\lim_{\tau \to 0} \langle \varphi_{\mathbf{k}}(\tau) \varphi_{\mathbf{k}'}(\tau) \rangle = (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}') \frac{H^2}{2k^3} \tag{7}$$

   (b)
$$\lim_{\tau \to 0} \langle \varphi_{\mathbf{k}}(\tau) \Pi_{\mathbf{k}'}(\tau) \rangle = (2\pi)^3 \delta^3 (\mathbf{k} + \mathbf{k}') \frac{1}{2k\tau} , \tag{8}$$

   (c)
$$\lim_{\tau \to 0} \langle \Pi_{\mathbf{k}}(\tau) \varphi_{\mathbf{k}'}(\tau) \rangle = \lim_{\tau \to 0} \langle \varphi_{\mathbf{k}}(\tau) \Pi_{\mathbf{k}'}(\tau) \rangle , \tag{9}$$

   (d)
$$\lim_{\tau \to 0} \langle \Pi_{\mathbf{k}}(\tau) \Pi_{\mathbf{k}'}(\tau) \rangle = (2\pi)^3 \delta^3 (\mathbf{k} + \mathbf{k}') \frac{k}{2H^2\tau^2} . \tag{10}$$

Extra: You could try to also show these in the Schrödinger picture, using the wavefunction (30).

4. *My first cosmological correlator*

   Compute the bispectrum of a massless scalar field on de Sitter induced by the interactions $\dot{\varphi}^3$ and $\dot{\varphi}(\partial_i\varphi)^2$.

5. *Wavefunction: the path integral formalism*

   In this problem you will compute the Gaussian wavefunction of a massless scalar in de Sitter and Minkowski spacetime using the path integral formalism. Recall that the wavefunction is defined by the path integral

$$\Psi[\bar{\phi};\eta_0] = \int_{\Omega}^{\Phi(\eta_0)=\phi} [D\Phi]\, e^{iS[\Phi]}\,, \tag{11}$$

   where the action is

$$S = -\frac{1}{2}\int d^3x\, d\eta\, a^4\, \partial_\mu\Phi\partial^\mu\Phi\,. \tag{12}$$

   (a) By a shift in the action or otherwise, show that $\Psi[\phi;\eta_0] = e^{iS[\Phi_{\rm cl}]}$ up to an irrelevant phase, where

$$\Phi_{\rm cl}(\mathbf{k},\eta) = \phi(\mathbf{k})\frac{f_k^*(\eta)}{f_k^*(\eta_0)}\,, \tag{13}$$

   and $f_k(\eta)$ are the mode functions.

   (b) Hence show that $\Psi[\phi,\eta]$ takes the form

$$\Psi[\phi;\eta_0] \equiv \exp\left[+\frac{1}{2}\int_{\mathbf{k},\mathbf{k}'}(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\psi_2(k)\phi(\mathbf{k})\phi(-\mathbf{k})\right]\,, \tag{14}$$

   where

$$\psi_2 = \frac{i}{H^2\eta_0^2}\frac{\partial_\eta f_k^*(\eta_0)}{f_k^*(\eta_0)}\,. \tag{15}$$

   Hint: you can show that $S[\Phi_{cl}]$ reduces to a boundary term by brute force, but it's much easier to integrate by part the kinetic term and drop the term proportional to the equations of motion.

   (c) Using the dS mode functions show that

$$\psi_2(k,\eta_0) = \frac{ik^2}{H^2\eta_0(1-ik\eta_0)} = \frac{1}{H^2}\left[-\frac{k^3}{(1+k^2\eta_0^2)} + i\frac{k^2}{\eta_0(1+k^2\eta_0^2)}\right]\,. \tag{16}$$

   Discuss this result: what is the sign of $\mathrm{Re}\,\psi_2$ and why does that matter? What happens to $\mathrm{Im}\,\psi_2$ in the late time limit and should we be worried about it?

6. **Optional:** *Factorized and commutator forms for in-in correlators*

   Prove that the "factorised" and "commutator" expressions,

$$\langle\mathcal{O}(\tau)\rangle = \langle 0|\left[\bar{T}e^{\left(i\int_{-\infty(1+i\delta)}^{\tau}d\tau'\hat{\mathcal{H}}_{\rm int}(\tau')\right)}\right]\hat{\mathcal{O}}(\tau)\left[Te^{\left(-i\int_{-\infty(1-i\delta)}^{\tau}d\tau'\hat{\mathcal{H}}_{\rm int}(\tau')\right)}\right]|0\rangle\,, \tag{17}$$

$$\begin{aligned}\langle\mathcal{O}(\tau)\rangle = \sum_{N=0}^{\infty} i^N\int_{-\infty}^{\tau}d\tau_N\int_{-\infty}^{\tau_N}d\tau_{N-1}\ldots\int_{-\infty}^{\tau_2}d\tau_1 \\ \times\langle 0|[\hat{\mathcal{H}}_{\rm int}(\tau_1),[\hat{\mathcal{H}}_{\rm int}(\tau_2),\ldots[\hat{\mathcal{H}}_{\rm int}(\tau_N),\hat{\mathcal{O}}(\tau)]\ldots]]|0\rangle\,,\end{aligned} \tag{18}$$

for a generic in-in correlator are indeed equivalent.

Hint: Proceed by induction. First prove that they are equivalent at order $N = 0$ and $N = 1$. Then, assuming that they agree at order $N - 1$, take the time derive of each Nth-order expression and rewrite it as the correlators of some other field to order $N - 1$. This proves that the expression agree to order N up to a constant. By taking the limit $t \to -\infty$ show that the constant has to vanish.

7. **Optional:** *Classical dynamics of a $P(X, \phi)$ scalar in FLRW*

Consider a $P(X, \phi)$ theory,

$$S[\phi] = \int d^4x \sqrt{-g} P(X, \phi) \,. \tag{19}$$

(a) Find the classical equation of motion for ϕ and its stress-energy tensor $T^{\mu\nu}$.

(b) By comparing with the $T^{\mu\nu}$ of a perfect fluid,

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + g_{\mu\nu} p \,, \tag{20}$$

identify ρ, p and u_μ in terms of ϕ, P and their derivatives.

(c) Show that your equation of motion can also be derived by combining the two Friedmann equations (i.e. the 00 and ii parts of the Einstein equations), which for a perfect fluid are,

$$3M_P^2 H^2 = \rho \,, \qquad\qquad -\dot{H} M_P^2 = \frac{1}{2} (\rho + p) \,. \tag{21}$$

8. **Optional:** *Massive scalar in de Sitter*

This question computes the power spectrum of a *massive* scalar field in de Sitter, described by the quadratic action,

$$S_2[\varphi] = \int d^4x \sqrt{-g} \left[-\frac{1}{2} \nabla_\mu \varphi \nabla^\mu \varphi - \frac{1}{2} m^2 \varphi^2 \right] \,. \tag{22}$$

(a) In the Heisenberg picture, $\hat{\varphi}_{\mathbf{k}}$ can be expanded in terms of creation and annihilation operators $\{\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}}\}$ and mode functions $f_k(\tau)$ as in (3). Derive the evolution equation that $f_k(\tau)$ has to satisfy from the action (22), using conformal time.

(b) To solve this equation, re-write it as an equation for $g_k = (-\tau)^{-3/2} f_k$, and then use the fact that the two linearly independent solutions of Bessel's differential equation,

$$x^2 \partial_x^2 y + x \partial_x y + (x^2 - \nu^2) y = 0 \,, \tag{23}$$

can be taken to be the two Hankel functions ($H_\nu^{(1)}$ and $H_\nu^{(2)}$). This should give you the most general solution for f_k, with two integration constants.

(c) Determine these integrations constants, either using the canonical commutation relations and Bunch-Davies vacuum condition or by matching this solution in the $-k\tau \to \infty$ limit to the Minkowski solution as in question 4. You should find

$$f_k(\tau) = \frac{\sqrt{\pi} H}{2} (-\tau)^{3/2} H_\nu^{(1)}(-k\tau) \quad \text{with} \quad \nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \,. \tag{24}$$

Hint: You may use the following expansions of the Hankel functions for $x \to \infty$

$$H_\nu^{(1)}(x) \simeq \sqrt{\frac{2}{\pi}} \frac{e^{ix}}{\sqrt{x}} \,, \qquad\qquad H_\nu^{(2)}(x) \simeq \sqrt{\frac{2}{\pi}} \frac{e^{-ix}}{\sqrt{x}} \,, \tag{25}$$

which are valid up to an irrelevant (ν-dependent) phase.

(d) Finally, compute the corresponding power spectrum (e.g. by considering $\langle \varphi_{\mathbf{k}}(\tau)\varphi_{\mathbf{k}'}(\tau)\rangle$ in the Heisenberg picture).

9. **Optional:** *Wavefunction: the canonical formalism*

The Hamiltonian for a free, massless scalar on an FLRW background is,

$$\mathcal{H}[\varphi, \Pi] = \int_{\mathbf{k}} \frac{1}{2} \left(\left| \frac{\Pi_{\mathbf{k}}}{a} \right|^2 + k^2 \left| a\, \varphi_{\mathbf{k}} \right|^2 \right) \tag{26}$$

where $\Pi_{\mathbf{k}}$ is the momentum conjugate to $\varphi_{\mathbf{k}}$.

(a) Show how this Hamiltonian follows from the action (1).

(b) Promoting $\varphi_{\mathbf{k}}$ and $\Pi_{\mathbf{k}}$ to operators, show that the annihilation operator,

$$\hat{a}_{-\mathbf{k}}(\tau_*) = \frac{1}{\sqrt{2k}} \left(\frac{i\hat{\Pi}_{\mathbf{k}}}{a(\tau_*)} + k a(\tau_*) \hat{\varphi}_{\mathbf{k}} \right) \tag{27}$$

can be used to diagonalise the Hamiltonian at time τ_*.

(c) Find the wavefunction $\psi(\tau_*) = \langle \varphi | 0 \rangle_{\tau_*}$ of the state $|0\rangle_{\tau_*}$ defined by

$$\hat{a}(\tau_*)|0\rangle_{\tau_*} = 0 \, . \tag{28}$$

Hint: the field-eigenstate $|\varphi\rangle$ behaves like a position-eigenstate from quantum mechanics, i.e.

$$\langle \varphi | \hat{\varphi}_{\mathbf{k}} | 0 \rangle_{\tau_*} = \varphi_{\mathbf{k}} \psi(\tau_*) \qquad \textit{and} \qquad \langle \varphi | \hat{\Pi}_{\mathbf{k}} | 0 \rangle_{\tau_*} = -i \frac{\delta}{\delta \varphi_{-\mathbf{k}}} \psi(\tau_*) \, . \tag{29}$$

(d) By making the Gaussian ansatz,

$$\psi(\tau) \;\propto\; \exp\left(- \int_{\mathbf{k}} \omega_k(\tau) \frac{\left| a(\tau) \varphi_{\mathbf{k}} \right|^2}{2} \right) \tag{30}$$

for the wavefunction at later times, show that the Schrödinger equation,

$$i \partial_\tau \psi = \mathcal{H} \left[\varphi, -i \frac{\delta}{\delta \varphi} \right] \psi \tag{31}$$

implies that the Gaussian width ω_k obeys,

$$i \partial_\tau \left(\omega_k a^2 \right) = (\omega_k^2 - k^2) a^2 \, , \tag{32}$$

with initial condition $\omega_k(\tau_*) = k$.

(e) Argue (using what you know about simple 1-dimensional Gaussian integrals) that the 2-point correlator in this state is,

$$\langle \hat{\varphi}_{\mathbf{k}} \hat{\varphi}_{\mathbf{k}'} \rangle = \frac{(2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}')}{2\, \mathrm{Re}\, (\omega_k a^2)} \, . \tag{33}$$